INTEGRAL KLEIN BOTTLE SURGERIES AND IMMERSED CURVES: SOME OBSTRUCTIONS FOR FIBERED KNOTS

1. Proof of $K = 6_2$ case

Determining $\widehat{HF}(S^3_4(6_2),[s])$, by finding $HF(\overline{\gamma_{6_2}}, \ell_4^{[s]})$, we see that when $K = 6_2$, the corresponding $\dim \widehat{HF}(S^3_4(6_2),[s]) = \begin{cases} 1 & \text{if } [s] = 0, 2. \\ 3 & \text{if } [s] = -1, 1. \end{cases}$

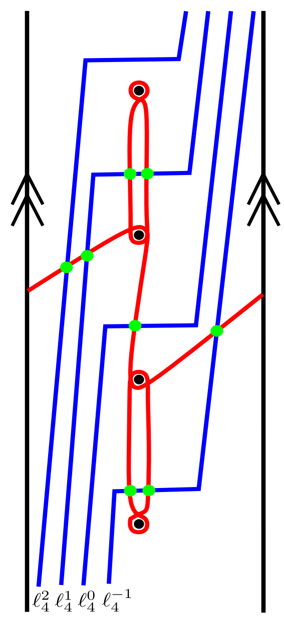

Figure 1. $\overline{\gamma_{6_2}}$ (red) vs $\ell_4^{[s]}$ (blue). The $\dim \widehat{HF}(S^3_4(6_2),[s])$ if $[s] = i$ is determined by the number of intersections of ℓ_4^i and $\overline{\gamma_{6_2}}$, as marked in green.

Theorem 1.1. *Suppose* $\dim \widehat{HF}(S^3_4(K),[s]) = \begin{cases} 1 & \textit{if } [s] = 0, 2. \\ 3 & \textit{if } [s] = -1, 1. \end{cases}$

and $J \subseteq S^3$. Then there does not exist a J such that $S^3_4(K) \cong (S^3 \backslash vJ) \cup_h \cdot N$.

Suppose, for the sake of contradiction, that there exists $J \subseteq S^3$ such that $S^3_4(K) \cong (S^3 \backslash vJ) \cup_h \cdot N$. Then there exists a configuration of J for which two of the curves in $\widetilde{\gamma}_N$ intersect $\widetilde{\gamma}_J$ three times, while the remaining two curves of $\widetilde{\gamma}_N$ intersect $\widetilde{\gamma}_J$ exactly once.

To restrict the possible configurations of J, we will analyze the behavior of $\widetilde{\gamma}_N$ and $\widetilde{\gamma}_J$ near the punctures. Since $\widetilde{\gamma}_N$ is fixed, its curves determine where intersections with any possible $\widetilde{\gamma}_J$ may occur. We distinguish the curves of $\widetilde{\gamma}_N$ by labeling them $\widetilde{\gamma}_{N_1}$, $\widetilde{\gamma}_{N_2}$, $\widetilde{\gamma}_{N_3}$, and $\widetilde{\gamma}_{N_4}$, listed from top to bottom.

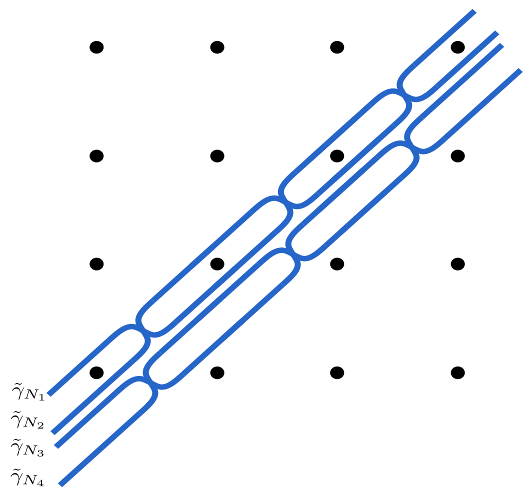

FIGURE 2. $\widetilde{\gamma}_N$ (blue). The four curves of $\widetilde{\gamma}_N$ are labeled as $\widetilde{\gamma}_{N_i}$ for $i = 1, 2, 3, 4$ from top to bottom.

One common behavior of $\widetilde{\gamma}_J$ is winding vertically around adjacent punctures (See Figure 3). When the curve is tightened, it behaves like a vertical line between the punctures, also known as the pegboard representation (See Figure 4). Combined with the fact that each curve of $\widetilde{\gamma}_N$ passes through specific vertical sections, this allows us to rule out certain $\widetilde{\gamma}_J$ whose vertical behavior would result in too many or too few intersections with $\widetilde{\gamma}_N$.

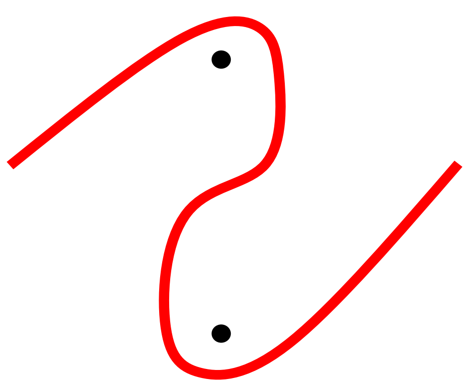

FIGURE 3. Non-tightened visualization of a common winding around two punctures for a possible $\widetilde{\gamma}_J$.

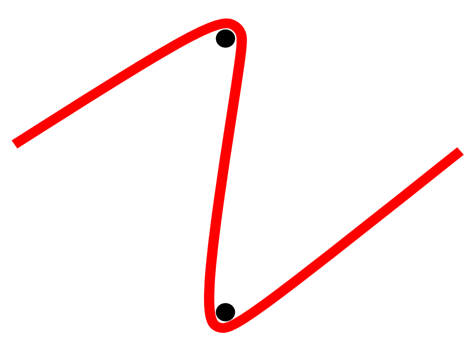

FIGURE 4. Tightened visualization of a common winding around two punctures for a possible $\widetilde{\gamma}_J$.

To describe the possible vertical behavior of $\widetilde{\gamma}_J$, we define n_k to be the number of vertical segments in the pegboard representative of $\widetilde{\gamma}_J$ that are parallel to $\overline{\mu}_k$, the lift of $\widetilde{\mu}$ at height k (see Figure 5). Outside of a small horizontal neighborhood around $\overline{\mu}$, possible intersections are less predictable; we denote this region by ρ.

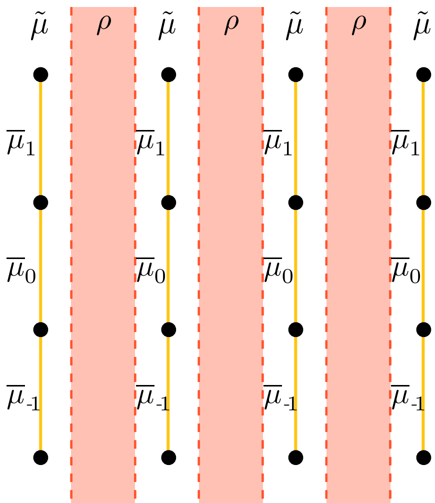

FIGURE 5. Vertical line $\overline{\mu}$ (orange), with $\overline{\mu}_k$ at height k, and the area outside of the vertical lines ρ (red).

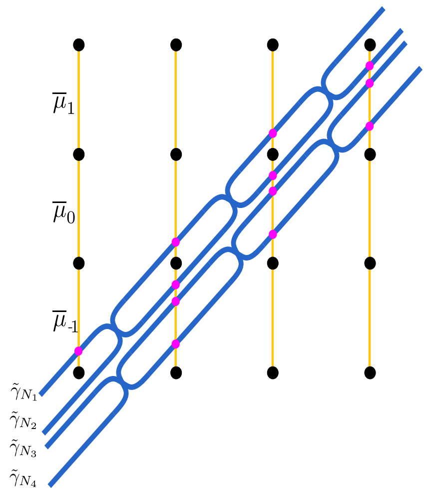

FIGURE 6. $\widetilde{\gamma_N}$ (blue) vs $\overline{\mu}_i$ (orange). Each intersection is marked with a purple dot.

For the reader's convenience, we introduce a simple notation to describe intersections involving $\widetilde{\gamma}_{N_i}$. We write $\pitchfork(\widetilde{\gamma}_{N_i}, \widetilde{\gamma}_J)$ to denote the number of intersections between $\widetilde{\gamma}_{N_i}$ and $\widetilde{\gamma}_J$, where each term appears with a coefficient indicating how many times it is intersected. For example, if $\widetilde{\gamma}_{N_x}$ intersects $\overline{\mu}_y$ once and $\overline{\mu}_z$ twice, and if $n_y = 1$ and $n_z = 1$, then:

$$\pitchfork(\widetilde{\gamma}_{N_x}, \widetilde{\gamma}_J) = n_y\overline{\mu}_y + 2n_z\overline{\mu}_z = \overline{\mu}_y + 2\overline{\mu}_z = 3$$

To avoid redundancy in our notation, we will write $n_k\overline{\mu}_k$ simply as n_k, since these terms typically appear together. Thus, for the example above:

$$\pitchfork(\widetilde{\gamma}_{N_x}, \widetilde{\gamma}_J) = n_y + 2n_z = 3$$

Furthermore, we must also account for intersections between $\widetilde{\gamma}_{N_i}$ and $\widetilde{\gamma}_J$ in the ρ region. Since these intersections cannot be determined explicitly without specifying the knot J, we introduce, for each $\widetilde{\gamma}_{N_i}$, a variable x_i representing the number of intersections between $\widetilde{\gamma}_{N_i}$ and $\widetilde{\gamma}_J$ in the ρ zone. Thus, if we set $\pitchfork(\widetilde{\gamma}_{N_i}, \overline{\mu}_k) = y_k$, then for any $\widetilde{\gamma}_{N_i}$ and $\widetilde{\gamma}_J$:

$$\pitchfork(\widetilde{\gamma}_{N_i}, \widetilde{\gamma}_J) = \cdots + y_{-1}n_{-1} + y_0n_0 + y_1n_1 + \cdots + x_i\rho$$

Using the pattern of $\widetilde{\gamma}_N$ shown in Figure 6, the intersections between $\widetilde{\gamma}_{N_i}$ and $\widetilde{\gamma}_J$ are given by:

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) &= \dots + 2n_{-3} + 2n_{-1} + 2n_1 + 2n_3 + \dots && +x_1\rho\\
\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) &= \dots + 2n_{-2} + 2n_0 + 2n_2 + \dots && +x_2\rho\\
\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) &= \dots + n_{-2} + n_{-1} + n_0 + n_1 + n_2 + \dots && +x_3\rho\\
\pitchfork(\widetilde{\gamma}_{N_4},\widetilde{\gamma}_J) &= \dots + n_{-2} + n_{-1} + n_0 + n_1 + n_2 + \dots && +x_4\rho
\end{aligned}$$

Lemma 1.2. *Suppose* $X = (S^3 \setminus \nu J) \cup N$*. If* $\dim(\widehat{HF}(X,[s]) \leq 3$ *for each* $[s]$*, then* $\forall k \in \mathbb{N}, \pitchfork(\widetilde{\gamma}_N, \overline{\mu}_{\pm k}) = 0$.

Since we know that $\widetilde{\gamma}_J$ is rotationally symmetric by π [HRW22, Theorem 7], then $n_k = n_{-k}$. Thus:

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) &= \dots + 2n_3 + 2n_1 + 2n_1 + 2n_3 + \dots && +x_1\rho\\
\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) &= \dots + 2n_2 + 2n_0 + 2n_2 + \dots && +x_2\rho\\
\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) &= \dots + n_2 + n_1 + n_0 + n_1 + n_2 + \dots && +x_3\rho\\
\pitchfork(\widetilde{\gamma}_{N_4},\widetilde{\gamma}_J) &= \dots + n_2 + n_1 + n_0 + n_1 + n_2 + \dots && +x_4\rho
\end{aligned}$$

And:

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) &= 4n_1 + 4n_3 + \dots && +x_1\rho\\
\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) &= 2n_0 + 4n_2 + \dots && +x_2\rho\\
\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) &= n_0 + 2n_1 + 2n_2 + \dots && +x_3\rho\\
\pitchfork(\widetilde{\gamma}_{N_4},\widetilde{\gamma}_J) &= n_0 + 2n_1 + 2n_2 + \dots && +x_4\rho
\end{aligned}$$

Since $\dim(\widehat{HF}(X,[s]) \leq 3$, we have that $\forall k \in \mathbb{N}, n_{\pm k} = 0$. Indeed, if $n_{\pm k} > 0$ for any odd integer k, then $\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) \geq 4$; similarly, if $n_{\pm k} > 0$ for any even integer $k > 0$, then $\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) \geq 4$. Therefore, $n_{\pm k} = 0$. Thus:

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) &= && x_1\rho\\
\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) &= 2n_0 && +x_2\rho\\
\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) &= n_0 && +x_3\rho\\
\pitchfork(\widetilde{\gamma}_{N_4},\widetilde{\gamma}_J) &= n_0 && +x_4\rho
\end{aligned}$$

From this, we also deduce that $n_0 < 2$, since if $n_0 \geq 2$, then $\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) \geq 4 > 3$. This leaves only a few possibilities for J. If we set $n_0 = 0$, the only possible J is the unknot (U); if we set $n_0 = 1$, the only candidates for J are the left-handed trefoil ($T(2,-3)$) and the right-handed trefoil ($T(2,3)$).

For the unknot, we find (Figure 7):

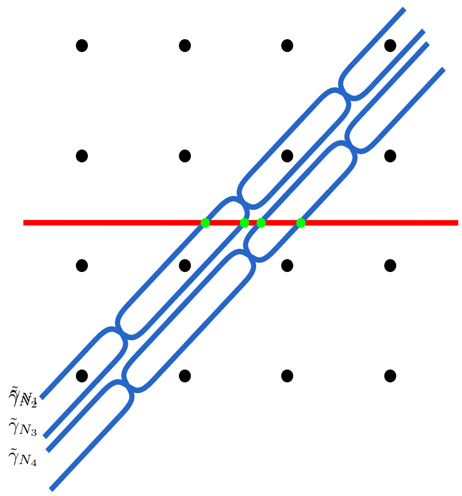

FIGURE 7. $\widetilde{\gamma}_U$ (red) vs. $\widetilde{\gamma}_N$ (blue). Intersections are marked in green.

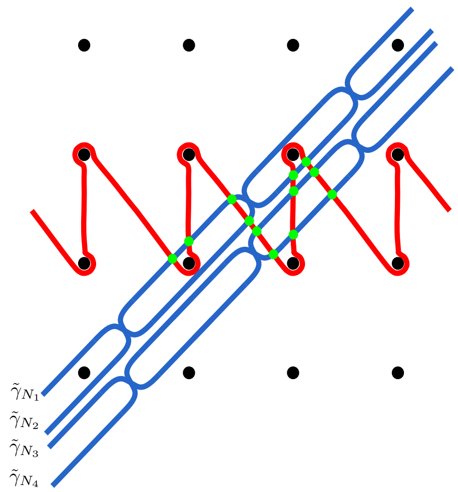

FIGURE 8. $\widetilde{\gamma}_{T(2,-3)}$ (red) vs. $\widetilde{\gamma}_N$ (blue). Intersections are marked in green.

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1}, \widetilde{\gamma}_J) &= 1 * \rho && = 1\\
\pitchfork(\widetilde{\gamma}_{N_2}, \widetilde{\gamma}_J) &= 1 * \rho && = 1\\
\pitchfork(\widetilde{\gamma}_{N_3}, \widetilde{\gamma}_J) &= 1 * \rho && = 1\\
\pitchfork(\widetilde{\gamma}_{N_4}, \widetilde{\gamma}_J) &= 1 * \rho && = 1
\end{aligned}$$

Since $\pitchfork(\widetilde{\gamma}_{N_i}, \widetilde{\gamma}_J) \neq 3$ for any $i = 1, 2, 3, 4$, it follows that $J \neq U$.

For the left-handed trefoil, we find (Figure 8):

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1}, \widetilde{\gamma}_J) &= 1 * \rho && = 1\\
\pitchfork(\widetilde{\gamma}_{N_2}, \widetilde{\gamma}_J) &= 2 * \overline{\mu}_0 + 3 * \rho && = 5\\
\pitchfork(\widetilde{\gamma}_{N_3}, \widetilde{\gamma}_J) &= 1 * \overline{\mu}_0 + 2 * \rho && = 3\\
\pitchfork(\widetilde{\gamma}_{N_4}, \widetilde{\gamma}_J) &= 1 * \overline{\mu}_0 + 2 * \rho && = 3
\end{aligned}$$

Since $\pitchfork(\widetilde{\gamma}_{N_2}, \widetilde{\gamma}_J) = 5$, it follows that $J \neq T(2, -3)$.

Finally, for the right-handed trefoil, we find (Figure 9):

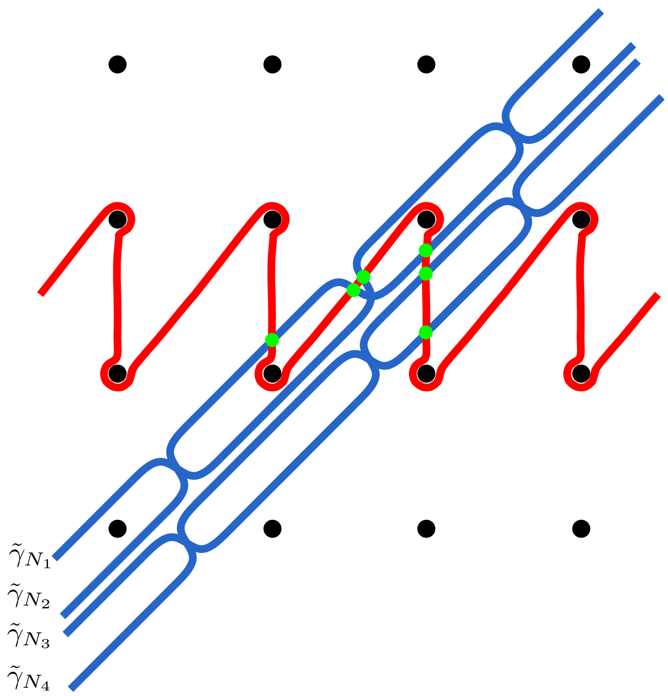

FIGURE 9. $\widetilde{\gamma}_{T(2,3)}$ (red) vs. $\widetilde{\gamma}_N$ (blue). Intersections are marked in green.

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1}, \widetilde{\gamma}_J) &= 1 * \rho && = 1\\
\pitchfork(\widetilde{\gamma}_{N_2}, \widetilde{\gamma}_J) &= 2 * \overline{\mu}_0 + 1 * \rho && = 3\\
\pitchfork(\widetilde{\gamma}_{N_3}, \widetilde{\gamma}_J) &= 1 * \overline{\mu}_0 && = 1\\
\pitchfork(\widetilde{\gamma}_{N_4}, \widetilde{\gamma}_J) &= 1 * \overline{\mu}_0 && = 1
\end{aligned}$$

Since only one curve results in 3 intersections, it follows that $J \neq T(2,3)$.

Thus, no $\widetilde{\gamma}_J$ exists that results in the correct number of intersections with the curves of $\widetilde{\gamma}_N$.

$\therefore \nexists J \ni S^3_4(6_2) \cong (S^3 \backslash vJ) \cup_h \cdot N$.

2. Proof of Fibered Knot case ($g(K) = 2$)

For the reader's convenience, we simplify the four possible values of $[s]$ in $\dim \widehat{HF}(S^3_4(K), [s])$ into a reduced notation involving only two values. This reduction is justified as follows. First, the cases $[s] = 1$ and $[s] = -1$ yield the same dimension, since $\overline{\gamma}_K$ is rotationally symmetric ([HRW22, Theorem 7]). Therefore, we treat these two cases as equivalent. Additionally, we have $\dim \widehat{HF}(S^3_4(K), [s]) = 1$ when $[s] = 2$, because for any knot K, the curve $\overline{\gamma}_K$ intersects ℓ^2_4 in exactly one location—specifically, the horizontal component on the left. Any other potential intersections at height 2 can be avoided when $\overline{\gamma}_K$ is tightened.

The notation we will adopt for $\dim \widehat{HF}(S^3_4(K),[s])$ is the ordered pair $\{\alpha,\beta\}$ where α is $\dim \widehat{HF}(S^3_4(K),[s])$ if $[s]=0$, and β is $\dim \widehat{HF}(S^3_4(K),[s])$ if $[s]=\pm 1$.

Theorem 2.1. *Suppose* $\dim \widehat{HF}(S^3_4(K),[s])$ *is determined by a Fibered Knot K,* $K \subseteq S^3$*, of genus 2* ($g(K)=2$) *and let* $J \subseteq S^3$ *such that* $S^3_4(K) \cong (S^3 \backslash vJ) \cup_h \cdot N$*. Then J or its curve invariant* $\overline{\gamma}_J$ *must satisfy:*

- $J = U$
- $J = T(2,\pm 3)$
- $J = 4_1$
- $\overline{\gamma}_J = \overline{\gamma}_U \bigcup \overline{\gamma}_{4_1}$

Definition 2.2. Fibered Knot: A Knot K_0 is fibered if at its genus height $g(K_0)$, only one intersection with the curve $\widetilde{\gamma}_{K_0}$ and $\overline{\mu}_{g(K_0)}$ exists.

Proposition 2.3. In the case where $K \subseteq S^3$ is a fibered knot of genus $g(K)=2$, the only distinct values of $\dim \widehat{HF}(S^3_4(K),[s])$ for which there exists a knot or link $J \subseteq S^3$ satisfying $S^3_4(K) \cong (S^3 \backslash vJ) \cup_h \cdot N$ are the following pairs: $\{1,1\},\{3,1\},\{5,3\},\{7,3\}$.

We can restrict the possible values of $\dim \widehat{HF}(S^3_4(K),[s])$ for which there may exist a knot or link $J \subseteq S^3$ such that $S^3_4(K) \cong (S^3 \backslash vJ) \cup_h \cdot N$ to the following eight distinct combinations: $\{1,1\},\{1,3\},\{3,1\},\{3,3\},\{5,1\},\{5,3\},\{7,1\},\{7,3\}$. We can put these limitations because:

$$\begin{aligned} i)&\text{Both } \alpha \text{ and } \beta \text{ must be odd.}\\ ii)&\ \max(\beta)=3.\\ iii)&\ \max(\alpha)=7.\end{aligned}$$

i) Previously established theorem?

ii) Since K is fibered, the possible configurations of $\overline{\gamma}_K$ at height 1 are constrained. These configurations depend on the value of the knot invariant τ_K:

(a) If $\tau_K = 0$ or $\tau_K = 1$, then a figure-eight curve must appear at height 1 in order to ensure that $g(K)=2$. However, only a single figure-eight is permitted; the presence of more than one would violate the fiberedness of K. In this case, $\dim \widehat{HF}(S^3_4(K),[s]) = 3$ for $[s]=\pm 1$, arising from one intersection with the $\widetilde{\gamma}_0$ curve and two intersections with the figure-eight.

(b) If $\tau_K = 2$, then $\overline{\gamma}_K$ begins at height 2 and descends to height -2 to maintain rotational symmetry. In this scenario, there can be at most one intersection with $[s]=\pm 1$ at height 1, occurring during the initial descent. Any attempt to return to height 2 would contradict the fiberedness of K. Moreover, no figure-eight can be centered at height 1, as it would intersect n_2, again violating the fibered condition. Consequently, we have $\dim \widehat{HF}(S^3_4(K),[s]) = 1$ for $[s] = \pm 1$.

Thus, $\max(\beta) = 3$.

Proposition 2.4. Let $k \in \mathbb{Z}$. Suppose $n_{|k|} = 0$, then it follows that $n_{|k|+1} = 2s$, for some $s \in \mathbb{N}$. Since $\widetilde{\gamma}_J$ is rotationally symmetric by π, the absence of vertical segments at at height $|k|$ implies that there is no vertical connection from $n_{|k|+1}$ to $n_{-(|k|+1)}$. The only structure that can exist in this setting is the figure-eight curve, each of which contributes two vertical segments at height $|k|+1$. Therefore, if $n_{|k|} = 0$, then $n_{|k|+1} = 2s$ where s is the number of figure-eight components centered at height $|k|+1$.

Proposition 2.5. Suppose $\dim \widehat{HF}(S^3_4(K),[s])$ is determined by a Fibered Knot K of genus 2 $g(K) = 2$, and let $J \subseteq S^3$ be such that$S^3_4(K) \cong (S^3 \backslash vJ) \cup_h \cdot N$. Then it must be the case that $n_k = 0$ for some $k \in \mathbb{N}$.

We already know that for any $\widetilde{\gamma}_N$ and $\widetilde{\gamma}_J$:

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) &= 4n_1 + 4n_3 + \dots && +x_1\rho \\
\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) &= 2n_0 + 4n_2 + \dots && +x_2\rho \\
\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) &= n_0 + 2n_1 + 2n_2 + \dots && +x_3\rho \\
\pitchfork(\widetilde{\gamma}_{N_4},\widetilde{\gamma}_J) &= n_0 + 2n_1 + 2n_2 + \dots && +x_4\rho
\end{aligned}$$

By Proposition 2.4, we can show that $n_k = 0, \forall k \in \mathbb{N} \ni k \geq 2$. Since every instance of $\dim \widehat{HF}(S^3_4(K),[s])$ includes the value 1 when $[s] = 2$, it follows that at least one curve of $\widetilde{\gamma}_N$ must intersect $\widetilde{\gamma}_J$ exactly once. Additionally, from the constraints discussed in (ii), two other curves of $\widetilde{\gamma}_N$ can intersect $\widetilde{\gamma}_J$ at most three times. In order for $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) = 1$ for some i, it is necessary that for all $k \geq 1$, both n_k and n_{k+1} cannot be simultaneously nonzero. Otherwise, the total intersection number would be $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) \geq 4,\ i = 1,2,3,4$. Therefore, for $k \geq 1$, if $n_{k+1} > 0, n_k = 0$. By Proposition 2.4, this implies that $n_{k+1} = 2s$ for some $s \in \mathbb{N}$. However, if $n_{k+1} > 1$, then $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) \geq 4, i = 3,4$, again violating the intersection constraints. Thus, for $k \geq 2, n_k = 0$:

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) &= 4n_1 && +x_1\rho \\
\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) &= 2n_0 && +x_2\rho \\
\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) &= n_0 + 2n_1 && +x_3\rho \\
\pitchfork(\widetilde{\gamma}_{N_4},\widetilde{\gamma}_J) &= n_0 + 2n_1 && +x_4\rho
\end{aligned}$$

Further, if $n_1 > 0$, then either (a) $n_0 \geq 1$, resulting in $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) \geq 2, i = 1,2,3,4$. Or (b) $n_1 > 1, n_0 = 0$, which, by Proposition 2.4, would cause $n_1 = 0 + 2s$ resulting in $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) \geq 4, i = 3,4$. Therefore, $n_k = 0$ for some $k \in \mathbb{N}/$:

$$\begin{aligned}
&\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) = && +x_1\rho\\
&\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) = 2n_0 && +x_2\rho\\
&\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) = n_0 && +x_3\rho\\
&\pitchfork(\widetilde{\gamma}_{N_4},\widetilde{\gamma}_J) = n_0 && +x_4\rho
\end{aligned}$$

iii) So far we have shown that the possibilities for each $\dim \widehat{HF}(S^3_4(K),[s])$ are:

$$\begin{aligned}
1 \text{ if } [s] &= 2\\
1,3 \text{ if } [s] &= \pm 1\\
1,3,5,7,9\dots \text{ if } [s] &= 0
\end{aligned}$$

Let $\alpha \geq 9$. Then $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) \geq 9$, while at least one of the other curves of $\widetilde{\gamma}_{N_i}$ results in $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) = 1$. By Proposition 2.5, we know:

$$\begin{aligned}
&\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) = && +x_1\rho\\
&\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) = 2n_0 && +x_2\rho\\
&\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) = n_0 && +x_3\rho\\
&\pitchfork(\widetilde{\gamma}_{N_4},\widetilde{\gamma}_J) = n_0 && +x_4\rho
\end{aligned}$$

To achieve a total intersection number $\pitchfork(\widetilde{\gamma}_N,\widetilde{\gamma}_J) \geq 9$, one would need to increase the value of n_0. However, n_0 must satisfy $n_0 < 4$, since if $n_0 \geq 4$, then $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) \geq 4, i = 3,4$, which would violate the constraints established in (ii). The only admissible configurations for J when $n_0 < 4$ are as follows:

$n_0 = 0$: The unknot U.
$n_0 = 1$: The trefoils $T(2,3), T(2,-3)$.
$n_0 = 2$: The unknot U with a figure-eight centered at height 0.
$n_0 = 3$: $T(2,3)$ or $T(2,-3)$ with a figure-eight centered at height 0.

Among these, the only configuration that results in 9 or more intersections is $T(2,-3)$ with a figure-eight centered at height 0. However, in this case $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) = 5$, for $i = 3,4$, which violates our constraints. Therefore, $\max(\alpha) = 7$.

Thus, the only possibilities for each $\dim \widehat{HF}(S^3_4(K),[s])$ are :

$$\begin{aligned}
1 \text{ if } [s] &= 2\\
1,3 \text{ if } [s] &= \pm 1\\
1,3,5,7 \text{ if } [s] &= 0
\end{aligned}$$

Which gives us the eight unique combinations of $\{1,1\}, \{1,3\}, \{3,1\}, \{3,3\}, \{5,1\}, \{5,3\}, \{7,1\}, \{7,3\}$.

Theorem 2.6. *Suppose* $\dim \widehat{HF}(S^3_4(K),[s]) = \{1,3\},\{5,1\},\{3,3\}$ *or* $\{7,1\}$. *Then* $\nexists J \subseteq S^3$ *such that* $S^3_4(K) \cong (S^3\backslash vJ) \cup_h \cdot N$.

1) Suppose $\dim \widehat{HF}(S^3_4(K),[s]) = \{1,3\}$. Then $\nexists J \ni S^3_4(K) \cong (S^3\backslash vJ) \cup_h \cdot N$.

See Theorem 1.1.

2) and 3) Suppose $\dim \widehat{HF}(S^3_4(K),[s]) = \{5,1\}$ or $\dim \widehat{HF}(S^3_4(K),[s]) = \{7,1\}$. Then $\nexists J$ such that $S^3_4(K) \cong (S^3\backslash vJ) \cup_h \cdot N$.

We know that, through proposition 2.5:

$$\begin{aligned}
&\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) = && +x_1\rho\\
&\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) = 2n_0 && +x_2\rho\\
&\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) = n_0 && +x_3\rho\\
&\pitchfork(\widetilde{\gamma}_{N_4},\widetilde{\gamma}_J) = n_0 && +x_4\rho
\end{aligned}$$

We cannot set $n_0 \geq 2$ as this would imply $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) \geq 2$ for $i = 3,4$ which violates the requirement that three curves of $\widetilde{\gamma}_{N_i}$ intersect $\widetilde{\gamma}_J$) only once. Therefore, we are restricted to the cases $n_0 = 1$ or $n_0 = 0$, which correspond to the following possibilities for J:

$n_0 = 0 : U$, for which $\pitchfork(\widetilde{\gamma}_{N_i},\widetilde{\gamma}_J) = 1$ for all $i = 1,2,3,4$.
$n_0 = 1 : T(2,3)$ and $T(2,-3)$. For which:
$\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) = 3$ for $T(2,3)$.
$\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) = 3$ for $T(2,-3)$.

None of these configurations yield the correct intersections patterns required for $\{5,1\}$ or $\{7,1\}$.

4) Suppose $\dim \widehat{HF}(S^3_4(K),[s]) = \{3,3\}$. Then $\nexists J$ such that $S^3_4(K) \cong (S^3\backslash vJ)\cup_h \cdot N$.

We know that, through proposition 2.5:

$$\begin{aligned}
&\pitchfork(\widetilde{\gamma}_{N_1},\widetilde{\gamma}_J) = && +x_1\rho\\
&\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) = 2n_0 && +x_2\rho\\
&\pitchfork(\widetilde{\gamma}_{N_3},\widetilde{\gamma}_J) = n_0 && +x_3\rho\\
&\pitchfork(\widetilde{\gamma}_{N_4},\widetilde{\gamma}_J) = n_0 && +x_4\rho
\end{aligned}$$

Since $n_0 < 2$ as any larger value would result in $\pitchfork(\widetilde{\gamma}_{N_2},\widetilde{\gamma}_J) \geq 4$, which violates the constraints, this again leaves us with either $T(2,3), T(2,-3)$, or U. However, none of these configurations yield $\{3,3\}$.

Therefore the only remaining dimension sets for which there exists a knot or link $J \subseteq S^3$ such that $S^3_4(K) \cong (S^3\backslash vJ) \cup_h \cdot N$ for a fibered knots K of genus $g(K) = 2$, and which have not been eliminated from consideration, are

$\{1,1\},\{3,1\},\{5,3\},\{7,3\}$. Their corresponding $\overline{\gamma}_K$ vs $\ell_4^{[s]}$ graphs are (Figure 10-13):

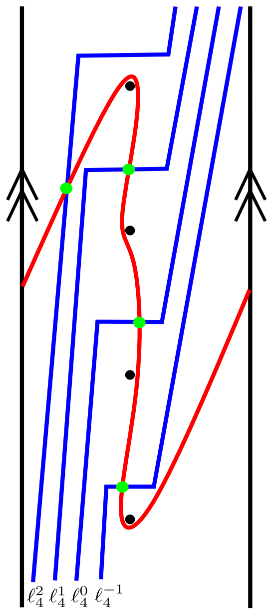

FIGURE 10. $\overline{\gamma}_K$ (red) vs $\ell_4^{[s]}$ (blue). Intersections are marked in green. Results in $\{1,1\}$.

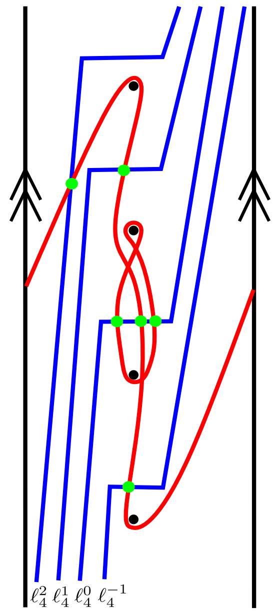

FIGURE 11. $\overline{\gamma}_K$ (red) vs $\ell_4^{[s]}$ (blue). Intersections are marked in green. Results in $\{3,1\}$.

By Proposition 2.5, we know that $n_k = 0$ for all $k \in \mathbb{N}$. Therefore, the only degrees of freedom in constructing possible $\widetilde{\gamma}_J$ arise from the value of n_0 and the structure of the horizontal component $\widetilde{\gamma}_{J_0}$. Since $n_0 \leq 4$, as any larger value would result in $\pitchfork(\widetilde{\gamma}_{N_i}, \widetilde{\gamma}_J) \geq 4, i = 3,4$, which is incompatible with our possible $\dim \widehat{HF}(S_4^3(K),[s])$. Thus, we restrict to the cases $n_0 = 0,1,2,3$, giving us the following knots:

$n_0 = 0$: The unknot U.

$n_0 = 1$: The trefoils $T(2,3), T(2,-3)$.

$n_0 = 2$: The unknot U with a figure-eight centered at height 0.

$n_0 = 3$: $T(2,3)$ or $T(2,-3)$ with a figure-eight centered at height 0.

Among these, five configurations correspond to the admissible dimension sets $\{1,1\},\{3,1\},\{5,3\},\{7,3\}$, and those are:

For $\{1,1\}$, $J = U$ (see Figure 7).

For $\{3,1\}$, $J = T(2,3)$ (see Figure 9).

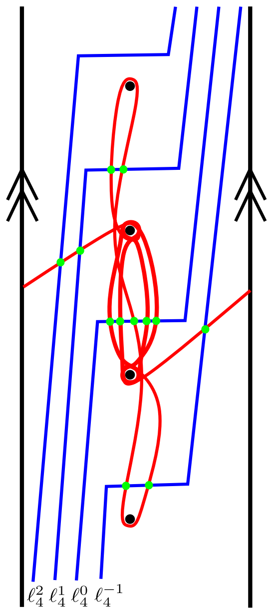

FIGURE 12. $\overline{\gamma}_K$ (red) vs $\ell_4^{[s]}$ (blue). Intersections are marked in green. Results in $\{5,3\}$.

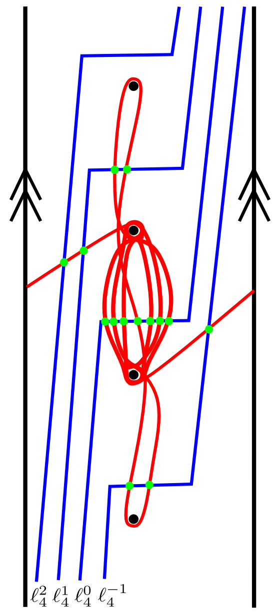

FIGURE 13. $\overline{\gamma}_K$ (red) vs $\ell_4^{[s]}$ (blue). Intersections are marked in green. Results in $\{7,3\}$.

For $\{5,3\}$, $J = T(2,-3)$ (see Figure 8) or $J = U \times figure - 8$ (see Figure 14) with :

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1}, \widetilde{\gamma}_J) &= 1 * \rho && = 1\\
\pitchfork(\widetilde{\gamma}_{N_2}, \widetilde{\gamma}_J) &= 4 * \widetilde{\mu}_0 + 1 * \rho && = 5\\
\pitchfork(\widetilde{\gamma}_{N_3}, \widetilde{\gamma}_J) &= 2 * \widetilde{\mu}_0 + 1 * \rho && = 3\\
\pitchfork(\widetilde{\gamma}_{N_4}, \widetilde{\gamma}_J) &= 2 * \widetilde{\mu}_0 + 1 * \rho && = 3
\end{aligned}$$

For $\{7,3\}$, $J = T(2,3) \times figure - 8$ (see Figure 15) with:

$$\begin{aligned}
\pitchfork(\widetilde{\gamma}_{N_1}, \widetilde{\gamma}_J) &= 1 * \rho && = 1\\
\pitchfork(\widetilde{\gamma}_{N_2}, \widetilde{\gamma}_J) &= 6 * \widetilde{\mu}_0 && = 7\\
\pitchfork(\widetilde{\gamma}_{N_3}, \widetilde{\gamma}_J) &= 3 * \widetilde{\mu}_0 && = 3\\
\pitchfork(\widetilde{\gamma}_{N_4}, \widetilde{\gamma}_J) &= 3 * \widetilde{\mu}_0 && = 3
\end{aligned}$$

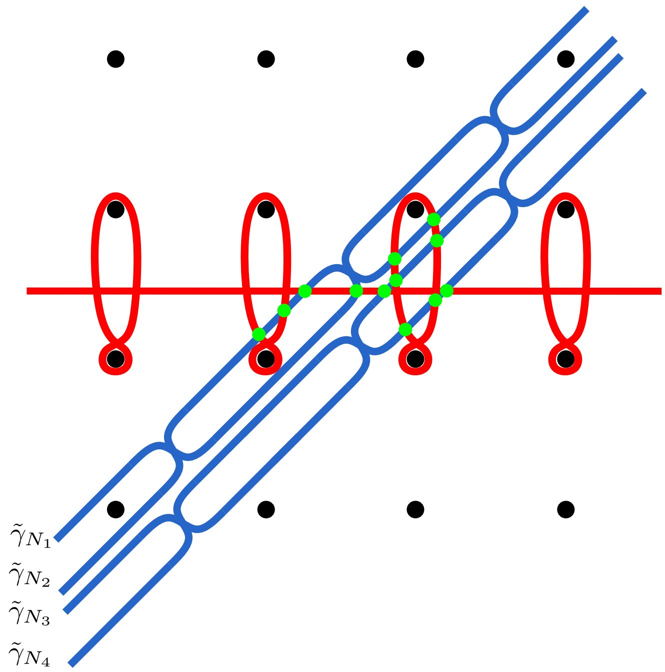

FIGURE 14. $\widetilde{\gamma}_?$ (red) vs. $\widetilde{\gamma}_N$ (blue). Intersections are marked in green. (Note: When tightened, the figure-8's centered at 0 behave as two intersections at each $\overline{\mu}_0$. Loosened for visual clarity.)

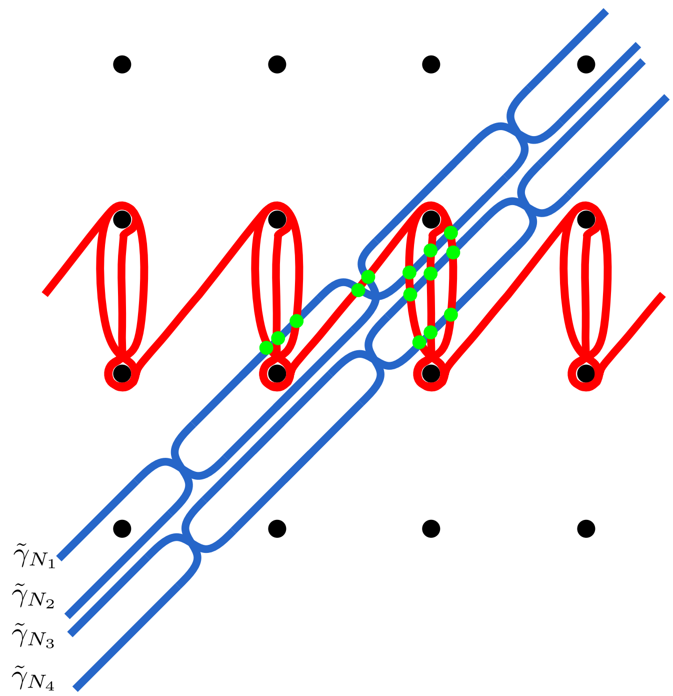

Figure 15. $\widetilde{\gamma}_{??}$ (red) vs. $\widetilde{\gamma}_N$ (blue). Intersections are marked in green.

References

[HRW22] Jonathan Hanselman, Jacob Rasmussen, and Liam Watson. Heegaard Floer homology for manifolds with torus boundary: properties and examples. *Proc. Lond. Math. Soc. (3)*, 125(4):879–967, 2022.